A note on the planar Skorokhod embedding problem

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Overview on Planar Brownian motion

Definition

A (one dimensional) Brownian motion starting at x, say $(B_t)_{t\geq 0}$, is a stochastic process with the following properties :

- $B_0 = x$.
- $t \mapsto B_t$ is a.e continuous.
- B_t has independent increments.
- $B_s - B_t \sim N(0, |s-t|)$, i.e normally distributed.

A planar Brownian motion $(Z_t)_t$ is simply the $2D$ process $(X_t, Y_t)_t$ where X_t and Y_t are two independent Brownian motion.

Overview on Planar Brownian motion

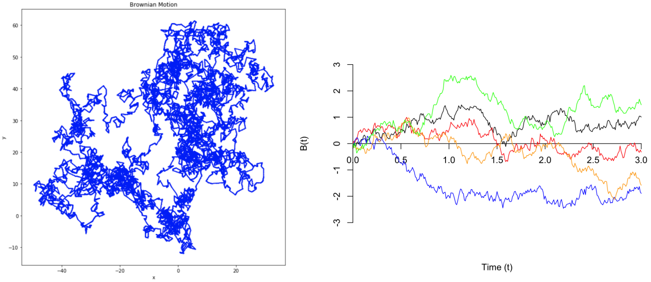

Figure 1: A Brownian path simulation

Overview on Planar Brownian motion

When the starting point is $Z_0 = 0$ then The probability density of Z_t is given by

$$p(t,z) = \frac{1}{(2\pi t)} e^{-\frac{|z|^2}{2t}}.$$

An important property of p is that it is a solution of the heat equation $(\partial_t - \frac{1}{2}\Delta)u = 0$. At $t = 0$, $p(0,z) = \delta_z$.

Killed Planar Brownian motion

In most cases, we need to stop (kill) Z_t at some instant τ, and to consider $(Z_t)_{t\leq\tau}$. A typical example of τ is the exit time from an open domain U, i.e

$$\tau_U := \inf\{t \mid Z_t \notin U\}.$$

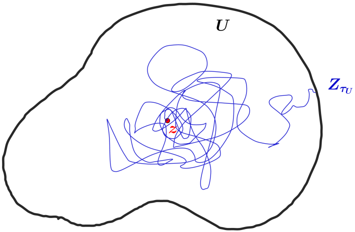

Figure 2: Stopped planar Brownian motion upon hitting the boundary of U

Links with PDE's

Run a planar Brownian motion inside a domain U (preferably simply connected). Then the following table highlights some of the connections between the corresponding stopped planar Brownian motion and the PDE theory.

PDE's	Solutions
Heat equation: $\begin{cases} u_t = \frac{1}{2}\Delta u \\ u(0,\cdot) = 1 \\ u(\cdot, \partial U) = 0 \end{cases}$	$\mathbf{E}_z(\mathbf{1}_{\{Z_{t\wedge\tau_U}\in U\}}) = \mathbf{P}_z(\tau_U > t)$
Torsion problem: $\begin{cases} \Delta u = -1 \\ u_{\mid\partial U} = 0 \end{cases}$	$\frac{1}{2}\mathbf{E}_z(\tau_U)$

Links with PDE's

PDE's	Solutions
Harmonic measure : $\begin{cases} \Delta u = 0 \\ u_{\mid \partial U} = 1_{E \subset \partial U} \end{cases}$	$\mathbf{P}_z(Z_{\tau_U} \in E)$

Example

If U is the unit disc, then

$$\mathbf{E}_z(\tau_U) = \frac{1 - |z|^2}{2}.$$

Links with PDE's

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- Monotonicity of the solutions against the monotonicity of the underlying domains.
- Dirichlet principal eigenvalue

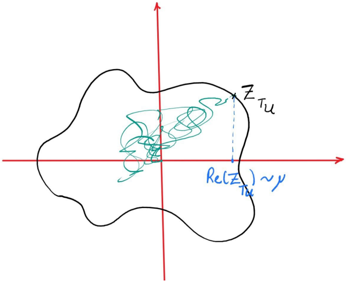
$$\lambda_U = -\frac{\ln(\mathbf{P}_z(\tau_U > t))}{t}$$

In particular (immediately)

$$U \subset V \Longrightarrow \lambda_V \leq \lambda_U.$$

Planar Skorokhod embedding problem

In [3] , the author considered the following question : Given a distribution μ with zero mean and finite second moment, is there a simply connected domain U such that if $Z_t = X_t + Y_t i$ is a standard planar Brownian motion, then $\Re(Z_\tau)$ has the distribution μ, where τ is the exit time from U?



Planar Skorokhod embedding problem

The answer is affirmative.

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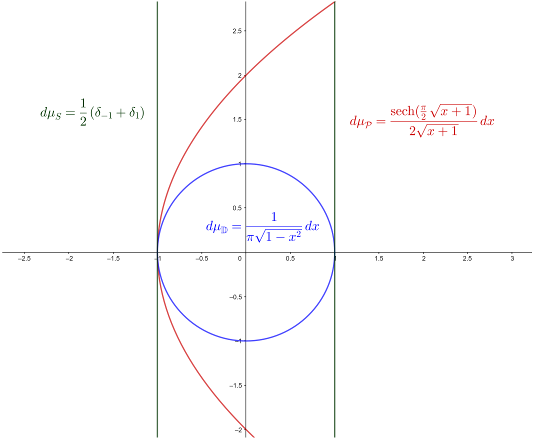

Figure 3: Examples of several constructions

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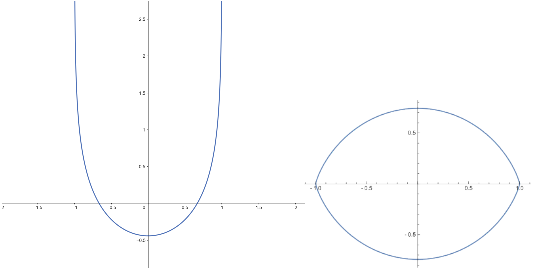


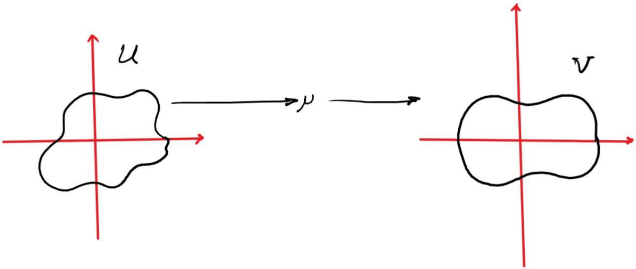
Figure 4: The two solutions generated out of the uniform distribution on $(-1, 1)$.

Brownian symmetrization

Let $\mathscr{B}$ denote the set of bounded simply connected domains (containing the origin). Then the following map is well defined

$$\mathfrak{B} : \begin{array}{l} \mathscr{B} \longrightarrow \mathscr{B} \\ U \longmapsto U_\mu \end{array}$$

where μ is the law of $\Re(Z_{\tau_U})$ and U_μ is the domain obtained by Gross' technique.



Optimization problems related to PSEP

Fix a probability distribution μ, say a bounded one. Among all domains solving the underlying PSEP, which one is extremal with respect to some property? In other words, we seek to find

$$\inf_{U:\mu\text{-domain}} \mathcal{J}(U)$$

where $\mathcal{J}$ is some functional.

References

[1] M. Boudabra and G. Markowsky. 2020. A new solution to the conformal Skorokhod embedding problem and applications to the Dirichlet eigenvalue problem. *J. Math. Anal. Appl.* 491, 2 (2020), 124351.

[2] M. Boudabra and G. Markowsky. 2020. Remarks on Gross' technique for obtaining a conformal Skorohod embedding of planar Brownian motion. *Electronic Communications in Probability* (2020).

[3] R. Gross. 2019. A conformal Skorokhod embedding. *Electronic Communications in Probability* (2019).